Pergamon

J. Appl. Maths Mechs, Vol. 61, No. 6, pp. 1025–1029, 1997

0021–8928/97 $24.00+0.00

PII: S0021–8928(97)001032-9

THE PRESSURE DISTRIBUTION AROUND A GROWING CRACK†

Yu. N. GORDEYEV and V. M. ENTOV

Moscow

(*Received* 17 *December* 1996)

Exact solutions of the problem of the pressure distribution around an ideal hydraulic fracture are derived. The crack propagates in a permeable porous medium following a square-root growth law. The case of the penetration of the fracturing fluid into a reservoir is also considered.

In studies of hydraulic fracturing the need arises to predict the transient fluid pressure field around a propagating hydrofracture [1, 2]. Assuming the reservoir rock and fluid to be elastically deformable and that the fluid flow in the reservoir outside the crack obeys Darcy's law, the pressure distribution is described by the heat conduction ("piezoconduction") equation [3, 4]

(0.1)

where the piezoconductivity k is of the order of 10^3–10^4 cm^2/s. To a first approximation the crack itself can be considered as an ideal one. Then it is a thin domain ("a surface") over which the pressure p^0 is specified, which is different from the initial reservoir pressure p_0. The problem is to determine the fracture growth (propagation) law and to obtain the perturbation of the reservoir pressure field caused by the crack propagation, and, of particular importance for analysing the hydraulic fracturing process, the distribution of the fracturing fluid leakage density (the seepage rate) over the crack surface. Generally, this problem must be solved numerically.

In this paper we consider several special cases (including those taking into account the displacement of the reservoir fluid by the fracturing fluid) which admit of a self-similar formulation and an exact solution and hence can be investigated fairly thoroughly. In particular, the solution of the problem of the propagation of a plane "ideal" crack (p_0 = const) in a permeable reservoir is obtained. The solutions may prove to be useful by themselves, and also for testing more universal numerical algorithms.

1. FORMULATION OF THE PROBLEM

It is required to find a solution of Eq. (0.1) in the domain outside the crack, assuming that a constant pressure p^0 is specified at the crack surface, while the pressure outside the crack is initially equal to the reservoir pressure p_0 = const, $p_0 < p^0$. The crack is modelled by a segment $|x| \leqslant l(t)$, $y = 0$ (case A, plane parallel flow) or by an infinitely thin disc, $r \leqslant l(t)$, $z = 0$ (case B, axisymmetric flow).

Case A corresponds to the propagation of a long vertical crack of rectangular shape; case B corresponds to the growth of a horizontal circular crack in a very thick reservoir.

Hence, we have the following problems

problem A:

$$\frac{\partial p}{\partial t} = k\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right), \quad -\infty < x < \infty, \quad y \geqslant 0$$

$$p(x,y,0) = p_0, \quad p(x,0,t) = p^0, \quad |x| \leqslant l(t); \quad \partial p / \partial y(x,0,t) = 0, \quad |x| > l(t)$$

†*Prikl. Mat. Mekh.* Vol. 61, No. 6, pp. 1060–1064, 1997.

problem B:

$$\frac{\partial p}{\partial t}=k\left(\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+\frac{\partial^2 p}{\partial z^2}\right),\quad 0<r<\infty,\quad 0<z<\infty$$

$$p(r,z,0)=p_0,\quad p(r,0,t)=p^0,\quad r\leqslant l(t);\quad \partial p/\partial z(r,0,t)=0,\quad r>l(t)$$

Here the obvious symmetry of the problem about the crack plane has been used. Owing to this the derivative of the pressure in a direction normal to the crack plane ("the flux") vanishes along the continuation of the crack.

It can be seen that the solutions of problems A and B are self-similar if the crack boundary propagates following a square-root law

$$L(t) = c\sqrt{t},\ c = \text{const}$$

Letting

$$x=c\xi\sqrt{t},\quad y=c\eta\sqrt{t},\quad r=c\rho\sqrt{t},\quad z=c\zeta\sqrt{t}$$

$$p=(p^0-p_0)f(\xi,\eta)+p_0\quad \text{if}\quad p=(p^0-p_0)\varphi(\rho,\zeta)+p_0$$

we obtain the problems for $f(\xi, \eta)$ and $\varphi(\rho, \zeta)$, respectively, which can be represented in a single form

$$\varepsilon\left(\frac{\partial^2\Phi}{\partial X^2}+\frac{\partial^2\Phi}{\partial Y^2}\right)+\left(2X+\frac{\varepsilon n}{X}\right)\frac{\partial\Phi}{\partial X}+2Y\frac{\partial\Phi}{\partial Y}=0,\quad \varepsilon=4\kappa c^{-2} \tag{1.1}$$

$$\Phi(X,0)=1,\ |X|\leqslant 1;\quad \partial\Phi/\partial Y(X,0)=0,\ |X|>1$$

$$\Phi(X,Y)\to 0,\quad (X^2+Y^2)^{1/2}\to\infty$$

$$\Phi=\begin{cases} f, & n=0\\ \varphi, & n=1\end{cases},\quad X=\begin{cases}\xi, & n=0\\ \rho, & n=1\end{cases},\quad Y=\begin{cases}\eta, & n=0\\ \zeta, & n=1\end{cases}$$

Here $n = 0$ for a plane crack, and $n = 1$ for an axisymmetric one.

2. THE SOLUTION OF PROBLEM (1.1)

In view of the obvious symmetry about the axes $x = 0$ and $y = 0$ we will solve the mixed boundary-value problem (1.1) in the domain $(0 \leqslant X < \infty) \cup (0 \leqslant Y < \infty)$. We introduce elliptic coordinates (u, v): $X = \operatorname{ch}u\cos v$, $Y = \operatorname{sh}u\sin v$ instead of (X, Y) coordinates; the respective coordinate lines form a system of confocal ellipses and hyperboles with foci at the points $(-1, 0)$ and $(1, 0)$. This transformation maps the domain $(0 \leqslant X < \infty) \cup (0 \leqslant Y < \infty)$ into the domain $(0 \leqslant u < \infty) \cup (0 \leqslant v < \pi/2)$.

In the elliptic coordinates the mixed boundary-value problem (1.1) reduces to the standard boundary-value problem in a half-strip

$$\varepsilon\left(\frac{\partial^2\Psi}{\partial u^2}+\frac{\partial^2\Psi}{\partial v^2}\right)+(\operatorname{sh}2u+\varepsilon n\operatorname{th}u)\frac{\partial\Psi}{\partial u}-(\sin 2v+\varepsilon n\operatorname{tg}v)\frac{\partial\Psi}{\partial v}=0 \tag{2.1}$$

$$\Psi(0,v)=1,\ 0\leqslant v\leqslant\frac{\pi}{2};\quad \frac{\partial\Psi}{\partial v}(u,0)=\frac{\partial\Psi}{\partial v}\left(u,\frac{\pi}{2}\right)=0,\quad 0<u,\quad \Psi(u,v)\to 0,\quad u\to\infty \tag{2.2}$$

where $\Psi(u, v) = \Phi(X, Y)$.

We will solve the problem using the method of separation of variables. The general solution of Eq. (2.11), satisfying the symmetry conditions, then has the form

$$\Psi(u,v)=\sum_{k=0}^{\infty}U_k(u)V_k(v) \tag{2.3}$$

where $U_k(u)$ and $V_k(v)$ are solutions of the equations

$$\varepsilon U_k'' + (\mathrm{sh}\,2u + \varepsilon n\,\mathrm{th}\,u)U_k' + k^2 U_k = 0,\ (\varepsilon\omega V_k')' - k^2\omega V_k = 0,\ \ \omega = \cos^n v\ \exp(\cos v\ /\ (2\varepsilon))$$

which satisfy the conditions

$$V_k'(0) = V_k'(\pi/2) = 0,\ \ U_k(u \to \infty) \to 0$$

The functions V_k form a complete orthonormal set of functions with weight ω over the segment $[0, \pi/2]$. We multiply the boundary condition at the crack surface (the first condition (2.2)) by ωV_k and integrate from 0 to $\pi/2$. We obtain

$$U_0(0) = 1,\ U_k = 0,\ k > 0.$$

Henceforth we will use the notation $U \equiv U_0(u)$, $V \equiv V_0(v)$ ($V_k(v) \equiv 0$ for $k \geqslant 1$). The required solution is found from the boundary-value problem

$$\varepsilon U'' + (\mathrm{sh}2u + \varepsilon n\mathrm{th}u)U' = 0;\ \ U(0) = 1,\ \ U(u \to \infty) \to 0 \tag{2.4}$$

The solution of (2.4) is expressed as

$$U(u) = \frac{\Lambda_n(u,\varepsilon)}{\Lambda_n(0,\varepsilon)},\ \ \Lambda_n(u,\varepsilon) = \int_u^\infty \exp\left(-\frac{\mathrm{ch}\lambda}{2\varepsilon}\right)\frac{d\lambda}{\mathrm{ch}\lambda} \tag{2.5}$$

The normalization constants $\Lambda_n(0, \varepsilon)$ can be expressed in terms of the McDonald functions $K_0(z)$ and $K_1(z)$ and the modified Struve functions $L_0(z)$ and $L_1(z)$ of the zeroth and first orders [5] for a plane crack ($n = 0$) and an axisymmetric crack ($n = 1$)

$$\Lambda_0(0,\varepsilon) = K_0\left(\frac{1}{2\varepsilon}\right)$$

$$\Lambda_1(0,\varepsilon) = \frac{\pi}{2} - \frac{1}{2\varepsilon}K_0\left(\frac{1}{2\varepsilon}\right) - \frac{\pi}{4\varepsilon}\left[K_0\left(\frac{1}{2\varepsilon}\right)L_1\left(\frac{1}{2\varepsilon}\right) - K_1\left(\frac{1}{2\varepsilon}\right)L_0\left(\frac{1}{2\varepsilon}\right)\right]$$

Thus, the solution of boundary-value problem (2.4) is given by the expression

$$\Phi(X,Y) = \Psi(u,v) = U(u) = \Lambda_n(u,\varepsilon)\ /\ \Lambda_n(0,\varepsilon) \tag{2.6}$$

$$u = \mathrm{arsh}\sqrt{\Delta_+},\ \ v = \arcsin\sqrt{\Delta_-},$$

$$\Delta_\pm = (\pm\Delta_0 + \sqrt{\Delta_0^2 + 4Y^2})/2,\ \ \Delta_0 = X^2 + Y^2 - 1$$

Expressing the derivative $\partial\Phi/\partial Y$ in elliptic coordinates taking into account equality (2.3), we find the leakage distribution over the crack edges and the total amount of fluid which penetrates from the fracture into the reservoir per unit time (the leakage flow rate) corresponding to solution (2.6)

$$\left.\frac{\partial\Phi}{\partial Y}\right|_{Y=0,0<X<1} = -\left.\frac{\Xi_n(\varepsilon)}{\sin v}\right|_{u=0,0<v<\pi/2} = -\frac{\Xi_n(\varepsilon)}{\sqrt{1-X^2}},\ \ \Xi_n(\varepsilon) = \frac{\exp(-1/(2\varepsilon))}{\Lambda_n(0,\varepsilon)}$$

$$Q = 4\pi^n \int_0^1 \left(-\left.\frac{\partial\Phi}{\partial Y}\right|_{Y=0,0<X<1} X^n\right)dX = 2^n\pi\Xi_n(\varepsilon) \tag{2.7}$$

It follows from the first relation of (2.7) that the distribution of the normal derivative of the pressure differs only by a normalizing factor from the distribution for a non-propagating crack, which is well known and corresponds to the point where the type of boundary conditions changes (the crack tip).

For small rates of crack propagation $\varepsilon \gg 1$ ($c \ll 1$), we have

$$\Xi_0(\varepsilon) \sim 1/\ln(4\varepsilon/\gamma),\ \ \Xi_1(\varepsilon) \sim 2/\pi \tag{2.8}$$

Here $\gamma = \exp(C)$ and $C \approx 0.7772$ is Euler's constant. Equation (2.8) shows that the normal derivative of the pressure at the edges of a plane crack vanishes as $\varepsilon \to \infty$. This is explained by the non-existence of a steady-state solution, bounded at infinity, corresponding to the flow from a crack in the plane case. One must either introduce a finite system size or take into account the transient nature of the process.

In the other limiting case, namely that of a high crack propagation rate, $\varepsilon \ll 1$ ($c \gg 1$), we have both for a plane crack ($n = 0$) and an axisymmetric crack ($n = 1$)

$$\Xi_n(\varepsilon) \sim 1/\sqrt{\pi\varepsilon} \tag{2.9}$$

This means the propagation of a self-similar pressure wave along the normal to the crack.

3. TAKING INTO ACCOUNT RESERVOIR FLUID DISPLACEMENT BY THE FRACTURING FLUID

Up to now it was implicitly assumed that the fracturing fluid and reservoir fluid are the same. Usually, however, the fracturing fluid has much greater viscosity than the reservoir fluid, and, on penetrating into the reservoir, it considerably changes the flow resistance in the vicinity of the crack. We will now consider the case of the displacement of the reservoir fluid by the fracturing fluid in the piston displacement approximation. For such flows both the equations and boundary conditions are identical with (1.1). However, the coefficient ε instead of being a constant is now a piecewise-constant function

$$\varepsilon = \begin{cases} \varepsilon_1 = (2\sqrt{k_1}/c)^2 & \text{(in the fracturing domain)} \\ \varepsilon_2 = (2\sqrt{k_2}/c)^2 & \text{(in the fracturing domain)} \end{cases} \tag{3.1}$$

The pressure and the normal component of the flux must be continuous at the domain boundary.

Generally speaking, this formulation leads to a complicated non-linear problem of the interaction of kinematic and heat waves. However, the simplicity of the solution guessed above enables us to hope that this problem also has a solution with an interface in the form of a growing ellipse, $u = u_0 = \text{const}$, where

$$\varepsilon(u) = \begin{cases} \varepsilon_1 = (2\sqrt{k_1}/c)^2, & u \leqslant u_0 \\ \varepsilon_2 = (2\sqrt{k_2}/c)^2, & u > u_0 \end{cases} \tag{3.2}$$

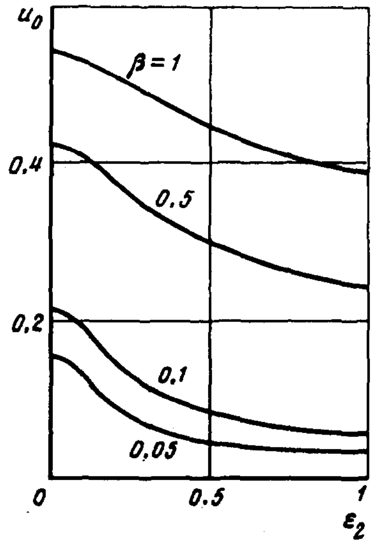

Fig. 1.

It turns out that such a solution does in fact exist. It is given by the expression

$$\Phi(X,Y) = \Psi(u,v) = U(u) = \Lambda_n(u,\varepsilon)/\Lambda_n(0,\varepsilon) \tag{3.3}$$

Solving the corresponding boundary-value problem (1.1), taking Eqs (3.2) into account by the separation-of-variables technique (cf. Section 2), we get

$$U(u) = \Lambda_n(u,\varepsilon)/\Lambda_n(0,\varepsilon) \tag{3.4}$$

$$\Lambda_n(u,\varepsilon) = \int_u^{u_0} \exp\left(-\frac{\operatorname{ch}\lambda}{2\varepsilon_1}\right)\frac{d\lambda}{\operatorname{ch}\lambda} + \int_{u_0}^{\infty} \exp\left(-\frac{\operatorname{ch}\lambda}{2\varepsilon_2}\right)\frac{d\lambda}{\operatorname{ch}\lambda}$$

The parameter u_0, which characterizes the interface between the fracturing fluid and the reservoir fluid, can be determined from the balance relation

$$2(2\pi)^n \int_0^t \int_0^l v_L(x,t)\,dx\,dt = mS;\quad v_L(x,t) = -\frac{2k^0}{\mu}\frac{\partial p(x,0,t)}{\partial y},\quad |x| \leqslant l(t) \tag{3.5}$$

Here $v_L(x, t)$ is the leakage rate of the fracturing fluid from the fracture into reservoir, m is the porosity and k^0 is the permeability of the reservoir rock, μ is the fracturing fluid viscosity and S is the area of the domain occupied by the fracturing fluid.

Equation (3.5) can be expressed in terms of dimensionless variables as

$$\Lambda_n(0,\varepsilon_1,\varepsilon_2) = \left(\frac{3}{2}\right)^n \varepsilon_1^2\beta\frac{\exp(-1/2\varepsilon_1))}{\operatorname{sh}(2u_0)\operatorname{ch}^n(2u_0)},\quad \beta = p_0(K_p^{-1} + K_m^{-1}) \tag{3.6}$$

where K_p and K_m are the bulk moduli of the fracturing fluid and of the porous medium, respectively.

Thus, the parameter $u_0 = u_0(\beta, \varepsilon_1, \varepsilon_2)$ is a root of the transcendental equation (3.6). The corresponding dependence is shown in Fig. 1 for $\varepsilon_1 = 1$ and $n = 0$.

This research was partially supported by the Russian Foundation for Basic Research (96-01-0748).

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Translated by the authors